

Deconfinement in the Two Dimensional XY Model

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The unbinding of vortex-antivortex pairs for the classical two-dimensional XY model in a magnetic field is studied. A single such pair is connected by a string of overturned spins, leading to linear confinement. We show that this system supports two phase transitions, one in which closed strings proliferate, and a second in which vortices unbind. The transitions are shown to be dual to one another, and are remarkably continuous. Possible consequences for a variety of systems are discussed.

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Introduction. One of the most studied [1] systems in condensed matter physics is the two-dimensional XY model,

$$H/k_B T = -K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) - h \sum_i \cos \theta_i. \quad (1)$$

In Eq. 1, θ_i represents the angle with respect to some reference direction of a planar spin located at lattice site i , $\sum_{\langle ij \rangle}$ denotes a sum over nearest neighbors, and h represents the effect of a magnetic field tending to align spins along $\theta_i = 0$. This model receives so much attention because it, and the closely related sine-Gordon model [2], describe a diverse range of two-dimensional classical, and one-dimensional quantum, systems. The model supports topological excitations in the form of vortices, which for $h = 0$ and high enough temperature unbind via a Kosterlitz-Thouless (KT) transition.

Surprisingly little work has been done to study the fate of this transition for $h > 0$. Early renormalization group (RG) studies [3] performed on the above model, with the last term replaced by $h \sum_i \cos(p\theta_i)$ with p an integer, indicated a range of parameters for which the term is irrelevant, presumably allowing a KT transition to be realized. For $p < 4$ the symmetry-breaking term is always relevant and the RG results are inconclusive. Nevertheless, for $p = 1$, the spins trivially exhibit long-range order, and it is commonly supposed that there can be no transition in this circumstance [4].

In this article, we demonstrate that vortex unbinding does indeed occur in the XY model with a magnetic field, although the nature of the transition is considerably changed from the KT behavior. In the absence of fluctuations, for large separations a state with a single vortex-antivortex pair will contain a string of reversed spins connecting them, leading to a linear pair potential [5]. In the absence of fluctuations, the width of this string is given by $\xi_0 = a_0 \sqrt{K/h}$, where a_0 is the underlying (square) lattice constant, and the energy per unit length is $8\sqrt{Kh}$ ($k_B T/a_0$) [6]. Even for small K , pairs will be tightly bound and relatively isolated if h and E_c , the vortex core energy, are large. As h and E_c decrease, vortices unbind in a two-step process: a first phase transition occurs in which the vortices remain bound in pairs, but the strings connecting them become unbounded in

length. The resulting state may be thought of as one in which closed strings have proliferated. This is followed by a transition in which the strings break open, with the endpoints being the vortices. This sequence has a dual description in terms of a solid-on-solid model with screw dislocations [7]. Interestingly, the vortex unbinding maps to a rough-flat transition (equivalently, domain wall proliferation) in the dual model, and string proliferation maps to dislocation unbinding. The transitions are remarkable in that the free energy is perfectly analytic as the system passes through them, in spite of a diverging length scale and a clear change in the low energy dynamics across the transition. Fig. 1 illustrates our proposed phase diagram for this system in the E_c vs. $1/h$ plane, for a fixed value of $K \approx 1/2\pi$.

Such deconfinement transitions have consequences for a variety of systems, some of which we will briefly describe towards the end of this article.

String Model. We begin by explaining why in a magnetic field, vortices come with strings attached. For this purpose, we write the continuum version of Eq. 1,

$$H = \int d^2 r \left\{ \frac{1}{2} K |\vec{\nabla} \theta(\vec{r})|^2 - h \cos \theta(\vec{r}) \right\}, \quad (2)$$

where we have chosen units in which $k_B T = 1$ and $a_0 = 1$. A vortex state is a local energy minimum of Eq. 2, subject to the constraint that $\int_P d\vec{r} \cdot \vec{\nabla} \theta = 2\pi n$, n an integer, along any closed path P surrounding the vortex core. Consider a state with a single vortex in a finite size system, and a large path surrounding it. Because of the field, nearly all the spins along the path must point in the preferred direction; the $2\pi n$ rotation will arise as a localized kink. If we choose the path parallel to $\vec{\nabla} \theta$ at the location of the kink and choose coordinates with the \hat{y} direction parallel to the path, $|\vec{\nabla} \theta(\vec{r})|^2 \approx (d\theta/dy)^2$, and we recognize the kink as the soliton of the sine-Gordon model [6]. Since this kink must appear for any large path enclosing the vortex, it is clear there is a string of overturned spins running from the vortex to the system boundary. The existence of such string defects in the context of the XY model in a magnetic field was noted in Ref. [8].

The unbinding transition for $h > 0$ is driven by fluctuations of the strings. To see this, we replace the partition

function for Eq. 1 by the Villain model, in which the substitution

$$e^{C \cos \theta} \rightarrow \sum_{s=-\infty}^{\infty} e^{-C(\theta-2\pi s)^2/2} \quad (3)$$

is made when cosines appear in the exponent. The Gaussian dependence of θ in the Villain model allows some progress in evaluating the partition function without sacrificing the critical properties of the XY model, since the two models share the same discrete periodic symmetry. Using the techniques of Ref. [3], the θ field may be integrated out, and with some manipulation [9] the partition function for the Villain model may be written as $Z_{VM} = \sum_{\{n(\mathbf{r}), A(\mathbf{r})\}} \exp[-H_{VM}]$, with

$$H_{VM} = \sum_{\mathbf{r}} \left[\frac{1}{2K} |\vec{\nabla} n(\mathbf{r}) + A(\mathbf{r}) \hat{x}|^2 + \frac{1}{2h} \left(\frac{\partial A}{\partial y} \right)^2 \right]. \quad (4)$$

In this expression $n(\mathbf{r})$ and $A(\mathbf{r})$ are integer degrees of freedom [10], and derivatives should be understood as really meaning discrete differences; e.g., $\partial A / \partial y \equiv A(\mathbf{r} + \hat{y}) - A(\mathbf{r})$.

It is useful to think of the n variables as residing at the centers of the square plaquettes formed by the lattice sites, and the A variables as residing on the nearest neighbor bonds in the \hat{y} direction. H_{VM} may then be thought of as a solid-on-solid model with screw dislocations [7]. In this interpretation, n is an integer height variable, and non-vanishing derivatives $\partial n / \partial x$, $\partial n / \partial y$ locate domain walls between different heights. The important low energy excitations involving $A \neq 0$ contain line segments along the domain walls in which $\partial n / \partial x + A = 0$, effectively removing part but not all of a domain wall (see Fig. 2). An endpoint of an “A” string – where $\partial A / \partial y \neq 0$ – may be identified as the core of a screw dislocation, and we see the second term in H_{VM} actually specifies its core energy to be $1/2h$. Our theory is thus effectively one in which the partition sum is over graphs containing open strings, as might be expected from the considerations discussed above.

Some comments are in order before we analyze H_{VM} . The model is closely related to one introduced in Ref. [3], which expresses the partition sum in terms of two interacting vortex fields. That model may be related to Z_{VM} by eliminating n using the Poisson summation formula, and representing the A sum in terms of $\partial A / \partial y$ [9]. The resulting Hamiltonian is identical to the vortex Hamiltonian of Refs. [3,4] (see Eq. 2.75b of Ref. [4]) with $E_c = 0$. A non-vanishing core energy may be introduced in that representation, after which the resulting theory enjoys a duality: upon interchanging $1/2h$ and E_c , and changing $K \rightarrow 1/4\pi^2 K$, the partition sum is identical to its original form. This observation will be important in obtaining the phase diagram.

RG Analysis. To make further progress we need a model with continuous fields but the same symmetries as

H_{VM} . A standard way to proceed [11] is to exchange the integer fields n and A for continuous variables ϕ and a , while adding terms that favor integer values of the fields. Taking the continuum limit, our effective Hamiltonian is

$$H_{eff} = \int d^2 r \left[\frac{1}{2K} |\vec{\nabla} \phi(\mathbf{r}) + a(\mathbf{r}) \hat{x}|^2 + \frac{\xi^2}{2K} \left(\frac{\partial a}{\partial y} \right)^2 - y \cos(2\pi \phi(\mathbf{r})) + y - y_a \cos(2\pi a(\mathbf{r})) + y_a \right]. \quad (5)$$

For large E_c , $y = e^{-E_c}$ is the usual vortex fugacity [12]. An important observation is that the low energy configurations of H_{eff} are highly analogous to those of H_{VM} , even for small y , y_a . Domain walls in n are analogous to soliton configurations of ϕ , and the a -field introduces local minima of H_{eff} that are analogous to open domain wall configurations. An endpoint of an open wall is analogous to a dislocation core in H_{VM} ; by matching the endpoint energy to the dislocation core energy $1/2h$, one may assign a value to y_a such that there is a reasonable mapping of the low energy configurations of H_{VM} to those of H_{eff} . A rough calculation yields $y_a \approx 1/16h$ [9].

Our analysis proceeds from the small y , y_a limit so that the cosine terms may be treated perturbatively. We wish to integrate out short length scale fluctuations in the partition function, which is accomplished by integration of $\phi(\mathbf{k})$, $a(\mathbf{k})$ in the shell $\Lambda/b < |k_{x,y}| < \Lambda$, where $b = e^l$ is the usual RG scale factor [11] and $\Lambda = \pi$ is the momentum cutoff. The scaling transformation takes the form $\mathbf{k} = \mathbf{k}'/b$, $\mathbf{r} = b\mathbf{r}'$, $\phi'(\mathbf{r}') = \phi(\mathbf{r})$, and $a(\mathbf{r}) = a'(\mathbf{r}')/b$. The renormalization of the $y_a \int d^2 r \cos[2\pi a(\mathbf{r})]$ term under this RG transformation is unusual and leads to the remarkable critical properties of the deconfinement transition. Since the argument of the cosine shrinks as the short wavelength fluctuations are integrated out, one is led to expand the cosine in a Taylor series,

$$y_a \cos(2\pi a) - y_a = \sum_{n=1}^{\infty} \frac{y_{2n}}{(2n)!} (-1)^n (2\pi a)^{2n}, \quad (6)$$

with the initial values $y_{2n}(l=0) = y_a$. Upon rescaling, power counting suggests $y_{2n} \sim b^{2-2n}$. Thus we expect all the terms to be irrelevant except the Gaussian one, which neither grows nor shrinks. The resulting fixed point Hamiltonian has the form

$$H_* = \int d^2 r \left[\frac{1}{2K} |\vec{\nabla} \phi(\mathbf{r}) + a(\mathbf{r}) \hat{x}|^2 + \frac{1}{2} \rho a(\mathbf{r})^2 \right]. \quad (7)$$

The parameter ρ plays an important role: it introduces a finite energy per unit length for creating the “a” strings that allow dislocation pairs to populate the system. Thus ρ is effectively a (renormalized) string tension, and for $\rho > 0$ dislocations are confined. The behavior of H_* reflects a qualitative difference when there is a finite

tension: for $\rho = 0$ there is a gapless mode for every value of k_x when $k_y \rightarrow 0$, whereas for $\rho > 0$ there is only a single gapless mode at $\mathbf{k} = 0$ [13]. As we will see, the RG flows determine which microscopic parameters allow a non-vanishing ρ .

In the vicinity of H_* , the scaling equations to lowest order in y , y_n become

$$\frac{dK}{dl} = 0 \quad (8)$$

$$\frac{d\xi^2}{dl} = -2\xi^2 \quad (9)$$

$$\frac{dy_{2n}}{dl} = -(2n-2)y_{2n} - 2\pi^2\mathcal{L}(\rho, \xi)y_{2n+2} \quad (10)$$

$$\frac{dy}{dl} = \left(2 - \eta\sqrt{K/\rho}\right)y. \quad (11)$$

The last of these equations has been evaluated at $\xi(l) = 0$, and η is a number of order unity. Note that $y_2 \equiv 4\pi^2\rho$, and the initial values $y_{2n}(l=0)$ are all given by y_a . The function \mathcal{L} arises from the momentum shell integral which may be evaluated exactly. For small values of ρ , it has the form

$$\mathcal{L} \approx \frac{K\Lambda^2}{\pi\sqrt{K\rho(1+\xi^2\Lambda^2)}}.$$

A remarkable feature of Eqs. 10 is that their solutions are related by $y_{2n+2}(l) = y_{2n}e^{-2l} \equiv y_{2n}\xi(l)^2/\xi_0^2$ where $\xi_0 = \sqrt{K/h}$ is the initial value of $\xi(l)$. Eqs. 10 can then be rewritten in terms of a single scaling equation,

$$\frac{d\rho}{d\xi^2} = \pi^2\mathcal{L}(\rho, \xi)\rho/\xi_0^2. \quad (12)$$

Fig. 3 illustrates the trajectories resulting from Eq. 12 for different initial values of ξ_0 . (The initial value of y_a may be written in terms of ξ_0 using the core energy matching discussed above.) For small values of ξ_0 , $\rho(l)$ lies below the separatrix (shown as a heavy line), touches the $\rho = 0$ axis at a finite value of $\xi^2/\xi_0^2 \equiv e^{-2l^*}$, and remains on this axis as it flows to the origin. We associate $l_{scr} = a_0e^{l^*}$ with a screening length: for separations below this the dislocations appear to be bound, while for larger separations the interaction is screened by other dislocations, allowing for an unbinding transition. This length scale diverges as the phase transition is approached. Above the separatrix, the flows end at a non-vanishing value of ρ , which we identify as a renormalized string tension: the dislocations are confined if ξ_0 is large enough. Thus, the separatrix represents a deconfinement line, with dislocations unbound if ξ_0 is smaller than some critical value ξ_{cr} .

Our RG analysis has demonstrated that for very large values of E_c , there is a phase transition in which *dislocations* unbind. The effect of *vortices* on this transition enters at second order in y , and one can show [9]

that this decreases the value of $\xi_0^2 = K/h$ at which the transition occurs as E_c decreases from infinity. On the other hand, at $E_c/K = 0$, $K/h = 0$, it may be shown that H_{VM} has the character of a set of decoupled one-dimensional solid-on-solid models [9], which should not have any phase transitions. Thus, we presume the dislocation unbinding line strikes the $K/h = 0$ axis at a finite value value of E_c/K , as shown in Fig. 1.

The dislocation confinement transition may be thought of as one in which domain walls go from open strings to proliferated closed loops. The confined phase is thus analogous to the rough phase of a solid-on-solid model [7,14]. For larger values of y (smaller E_c), a second transition to the smooth phase may occur [15]. In the dual of H_{VM} , this transition is equivalent to vortex unbinding. Because of the duality in the partition sum, this transition has precisely the same character as the dislocation unbinding transition, and we expect for large values of K/h that it should occur when $E_c/2\pi^2K = \xi_{cr}^2$; i.e., at the same value one finds for dislocation unbinding. This is the origin of the lower transition line in Fig. 1. It is interesting to note that the confined phase is actually one in which closed strings (loops) are proliferating. In the dislocation language, they are the domain walls associated with the height n ; in the dual representation (in which the vortex degrees of freedom are explicit) they correspond to strings of overturned spins.

A remarkable property of the deconfinement transition is that it is perfectly continuous. The “accumulation point” at $\rho = 0$, $\xi^2 = 0$ in Fig. 3 appears at the end of a line of fixed points, but there is no relevant direction (in the RG sense) leading away from it. Such relevant directions are the source of nonanalyticity in the free energy in standard second order phase transitions [11]. Experimentally, this implies that thermodynamic measurements cannot detect the deconfinement transition – one must instead directly probe the dislocations, vortices, or the correlation functions they affect.

Finally, we note that measurements of the vortex diffusion constant in XY model simulations have confirmed that vortex unbinding indeed occurs in this system [16].

Applications. The XY system in a magnetic field and several closely related models may be used to describe a large number of systems, and the transition discussed in this work has consequences for most of them. These include bilayer thin-film superconductors, for which the *staggered* conductivity would be controlled by a deconfinement transition; bilayer crystals, for which dislocations are analogous to vortices of the XY model, and deconfinement may lead to an unlocking transition of the layers; and double layer quantum Hall systems at total filling factor $\nu = 1$ [5], where vortices carry physical charge $\pm e/2$ and deconfinement may lead to charge fractionalization. One dimensional quantum systems (Luttinger liquids) are also often described by models closely related to Eqs. 1 and 2 [2], and the transitions described

in this work have consequences under a variety of circumstances. Finally, generalizations of the physics discussed here to layered systems, including vortices in high temperature superconductors, highly anisotropic crystals, and quantum fluctuations in 2+1 dimensional striped systems, are possible and are likely to lead to a variety of new phenomena.

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 - [13] The apparent broken $\pi/2$ rotational symmetry is an artifact of dislocation core energies depending on whether the domain wall (steps in n) exits the core along the x or y direction. A less physical but qualitatively similar model can be formulated that respects the symmetry [9].
 - [14] An interesting interpretation of this phase is that, in the dual of H_{VM} , vortices are *logarithmically* confined [9]. To be consistent this means that the dislocations in H_{VM} are also logarithmically confined in this phase.
 - [15] Eq. 11 suggests that H_{eff} supports a KT transition for large enough values of ρ . It is unlikely, however, that

H_{VM} accesses this transition, because the constraints of duality suggest a minimum value of E_c is required for vortex unbinding, in contrast to a KT transition.

[16] J.P. Straley and H.A. Fertig, unpublished.

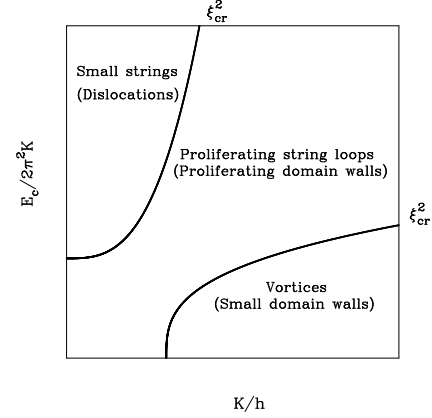


FIG. 1. Schematic phase diagram for XY model in a magnetic field for $K \approx 1/2\pi$. Upper left corner represents an ordered phase (unbound screw dislocations in dual representation). Middle phase contains proliferated loops in both descriptions, and lower right corner contains unbound vortices (flat phase in dual representation.)

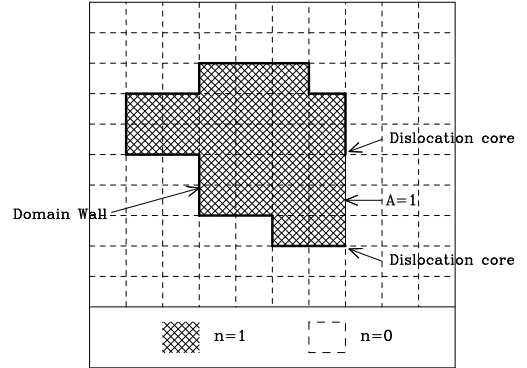


FIG. 2. A low energy configuration involving $A \neq 0$. A region of $n = 1$ (hatched squares) is embedded in a surrounding $n = 0$ region (white squares); heavy line represents a domain wall. The line segment with $A = 1$ cancels the domain wall energy for part of its length, leaving an open domain wall.

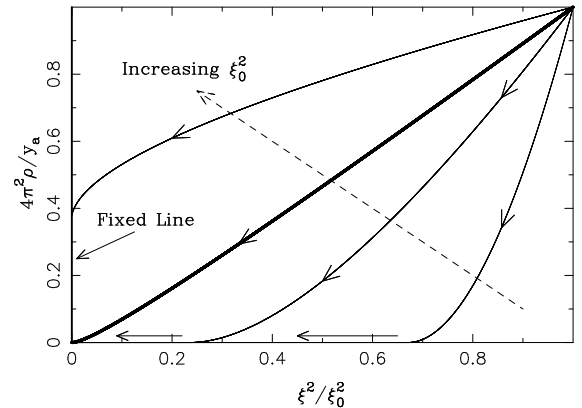


FIG. 3. RG flows for Eq. 12. Left vertical axis is a fixed line, and heavy line is a separatrix between flows that reach $\rho = 0$ for finite l and flow to the origin as $l \rightarrow \infty$, and those that have $\rho > 0$ at the end of their flow. Finite values of ρ in H_* indicate confinement.